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Large-scale instabilities of turbulent wakes

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The equations describing the statistical features of small amplitude waves in a turbulent shear flow are derived from the Navier–Stokes equations. Closure is achieved through a postulated constitutive equation for the alteration of the statistical properties of the turbulence by the organized wave. The theory is applied in an examination of the stability of a hypothetical wake consisting of small-scale turbulence enclosed within a steady uncontorted superlayer. A set of superlayer jump conditions is derived from fundamental considerations, and these are of more general interest. For this hypothetical flow the analysis predicts large-scale instabilities and superlayer contortions reminiscent of large-eddy structures observed in real flows. These instabilities therefore offer an explanation of the presence of large-scale organized motions in turbulent free shear flows.

1. Introduction

Turbulent shear flows are known to possess a rather high degree of organization. The large-eddy structure in wake flows has been studied in detail by Townsend (1956), Grant (1958) and, more recently, by Payne & Lumley (1967). Kovasznay, Kibens & Blackwalder's (1970) measurements in the wake region of a turbulent boundary layer give added insight into the structure near the convoluting superlayer. The origin of the large-scale structures is not well understood. Townsend (1966) sought an explanation in terms of a Kelvin–Helmholtz instability of a presumed vorticity concentration at the superlayer, but the selective-sampling measurements of Kovasznay *et al.* indicate that no such vorticity concentration exists.

The objective of the present paper is to show that a turbulent wake structure confined within a smooth uncontorted superlayer would be dynamically unstable to two- and three-dimensional wave-like disturbances. The Kelvin–Helmholtz instability arises because of a vorticity maximum *within* the wake, rather than a vorticity maximum at the superlayer. These instabilities give rise to the organized large-eddy structures, the amplitudes of the large-scale motions being determined by nonlinear interactions between such motions.

2. The governing equations

We require a closed set of dynamical equations for the velocity components of an organized wave superposed on a turbulence field. The development here is abbreviated; see Hussain & Reynolds (1970*b*) for more detail.

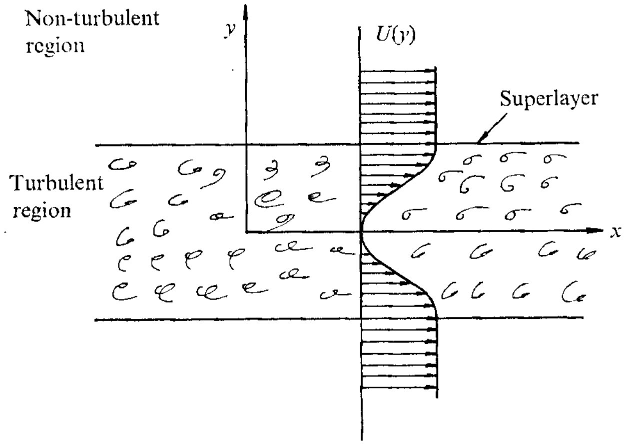

FIGURE 1. The undisturbed flow field.

In order to sort out an organized wave disturbance from a turbulent velocity field it is convenient to split the velocity into three parts:

$$u_i = \overline{u}_i + \tilde{u}_i + u_i'. \tag{2.1}$$

Here $\overline{u}_i$ is the time-averaged component, obtained in the conventional manner, u_i' the turbulence, and $\tilde{u}_i$ the organized wave. The wave component can be identified by phase averaging the total signal. Taking an average of the signal at a given phase in the cycle of the basic wave, and denoting this average by $\langle\ \rangle$, we have

$$\langle u_i \rangle = \overline{u}_i + \tilde{u}_i. \tag{2.2}$$

The dynamical equation for the organized wave is found from the Navier Stokes equation by introducing (2.1), phase averaging and subtracting the time average. On neglecting quadratic terms in $\tilde{u}_i$, this results in

$$\partial \tilde{u}_i / \partial x_i = 0, \tag{2.3a}$$

$$\frac{\partial \tilde{u}_i}{\partial t} + \overline{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} + \tilde{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{\partial \tilde{p}}{\partial x_i} + \frac{1}{R} \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j} - \frac{\partial \tilde{r}_{ij}}{\partial x_j}, \tag{2.3b}$$

where

$$\tilde{r}_{ij} = \langle u_i' u_j' \rangle - \overline{u_i' u_j'}. \tag{2.3c}$$

Here we have normalized using suitable characteristic length and velocity scales δ and U_r; $R = U_r \delta / \nu$. For a discussion of the equations for the energies $\overline{\tilde{u}_i \tilde{u}_i}$ and $\overline{u_i' u_i'}$ see Reynolds & Hussain (1972). The term $\tilde{r}_{ij}$ as defined in (2.3c) represents the oscillation in the background Reynolds stress produced by the passage of the organized wave. The closure problem presents itself in the determination of $\tilde{r}_{ij}$, for which some additional information must be given.

Our objective is to consider a turbulent wake surrounded by an uncontorted superlayer, i.e. a wake initially devoid of large-scale eddy structures (see figure 1). The stability analysis will then suggest a mechanism by which large-scale disturbances arise and are maintained. Hence, $\tilde{u}_i$ will represent the large-scale motions and u_i' the small-scale turbulence. It can be argued from reasonably fundamental grounds (see, for example, Lumley 1967a, b, 1970) that in the limit of weak slowly changing large-scale deformations the small-scale structure acts like a viscous fluid, i.e. it possesses an eddy viscosity. With due reservations we set

$$\langle u_i' u_j' \rangle = \tfrac{1}{3} q^2 \delta_{ij} - 2\epsilon(\bar{S}_{ij} + \tilde{S}_{ij}), \tag{2.4}$$

where

$$q^2 = \langle u_i' u_i' \rangle,$$

$$\bar{S}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \quad \tilde{S}_{ij} = \frac{1}{2}\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right).$$

Here ϵ is the 'eddy viscosity' of the small-scale turbulence. Now, if we further assume that the periodic distortions do not alter the turbulence energy,† but merely serve to alter the isotropy of the small-scale structure, it follows that

$$\tilde{r}_{ij} = -2\epsilon \tilde{S}_{ij}. \tag{2.5}$$

This is our closure model. In addition we shall neglect viscous forces and assume that ϵ is uniform over the region within the superlayer and zero outside the superlayer.

As in conventional stability theory, we shall treat the basic flow as quasi-parallel, with $\bar{u}_i = [U(y), 0, 0]$. The dynamical equations then admit eigensolutions of the form

$$\tilde{u}_i = \hat{u}_i(y) \exp[i(\alpha x + \beta z - \alpha c t)]. \tag{2.6}$$

We use $(x_1, x_2, x_3) = (x, y, z)$. The resulting equations for $\hat{u}_i$ may be combined and reduced using the Squire transformation to a single fourth-order equation, which with our assumptions is precisely the Orr–Sommerfeld equation of conventional stability theory (Lin 1955):

$$\{(D^2 - k^2)^2 - i\alpha R_\epsilon[(U-c)(D^2-k^2) - U'']\}\Phi = 0. \tag{2.7}$$

Here $\Phi(y)$ is the disturbance (u_2 or stream function) amplitude, $k^2 = \alpha^2 + \beta^2$, k being the (normalized) total wavenumber, $D = d/dy$, c is the (normalized) wave speed and $R_\epsilon = U_r \delta/\epsilon$ is the Reynolds number based on ϵ. Equation (2.7) will be used to describe the disturbance eigenfunctions in the turbulent region enclosed by the uncontorted superlayer.

The organized motions in the non-turbulent region outside the superlayer will be described by the inviscid form of (2.7):

$$\{(U-c)(D^2-k^2) - U''\}\Phi = 0. \tag{2.8}$$

3. Superlayer jump conditions

In order to connect the solutions in the turbulent and non-turbulent regions properly, coupling conditions across the superlayer must be derived. The conditions are independent of the constitutive model and are of considerable interest in their own right.

The superlayer coupling conditions will be developed by considering a thin control volume about a segment of the layer (figure 2), treating the superlayer in a quasi-steady manner. We shall denote the (dimensional) quasi-steady mean

† Perhaps realistic for rapid distortions.

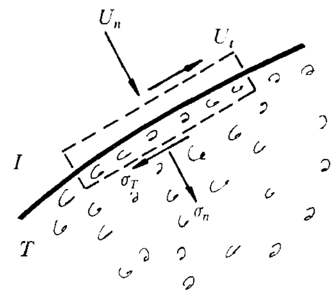

FIGURE 2. Control volume for derivation of the superlayer jump conditions.

velocity field by $(U_n, U_t, 0)$, where U_n and U_t are the components normal and tangential to the local superlayer surface. This mean field may possess a vorticity component Ω_p in the plane of the superlayer perpendicular to the n, t plane:

$$\Omega_p = \frac{\partial U_t}{\partial x_n} - \frac{\partial U_n}{\partial x_t}.$$

The n, t and p axes then form a locally orthogonal system, the t, p plane being the superlayer. The n axis is considered to be directed inwards towards the turbulent fluid. The subscripts I and T will denote the irrotational and turbulent regions respectively. Viscous stresses on either side will be neglected.

For incompressible flow, a mass balance gives

$$U_{nI} = U_{nT}. \tag{3.1a}$$

The normal momentum balance gives

$$P_I = (P + \overline{\rho u_n'^2})_T, \tag{3.1b}$$

where P denotes the mean pressure and $\overline{\rho u_n'^2}$ represents the turbulent normal stress exerted on the control volume. The tangential momentum balance gives

$$\rho U_n(U_{tI} - U_{tT}) - (\overline{\rho u_n' u_t'})_T = 0. \tag{3.1c}$$

The term $-\overline{\rho u_n' u_t'}$ represents the turbulent shearing stress on the superlayer; if this stress is not zero then the mean (quasi-steady) velocity component U_t tangential to the layer must exhibit a discontinuity across the layer. These three equations can alternatively be derived by integration of the appropriate differential equations across the superlayer.

Equations (3.1) do not form a complete set of coupling conditions. If one assumes $U_n = 0$ (which neglects entrainment), then from (3.1*c*) the layer can support no stress, as Townsend (1966) assumed. With this additional assumption (3.1*a*–*c*) yield Townsend's coupling conditions.

A fourth superlayer equation can be derived by integration of the vorticity equation across the superlayer, and with this additional equation the coupling is complete. The equation for Ω_p is (Townsend 1956)

$$\frac{\partial \Omega_p}{\partial t} + \frac{\partial}{\partial x_n}(U_n \Omega_p) + \frac{\partial}{\partial x_t}(U_t \Omega_p) + \frac{\partial}{\partial x_n}\overline{u_n' \omega_p'} + \frac{\partial}{\partial x_t}\overline{u_t' \omega_p'}$$
$$= \frac{\partial}{\partial x_n}\overline{u_p' \omega_n'} + \frac{\partial}{\partial x_t}\overline{u_p' \omega_t'} + \nu \nabla^2 \Omega_p,$$

where ω_p' is the fluctuation vorticity, $\omega_p' = \partial u_t'/\partial x_n - \partial u_n'/\partial x_t$, and no summation is implied. Both the mean and fluctuating vortices are zero in the irrotational external flow. Let us integrate this equation across the superlayer in the n direction. If we assume that the structure is locally homogeneous in the plane of the superlayer, the $\partial/\partial x_t$ terms do not contribute. Then, neglecting the viscous diffusion away from the superlayer, i.e. at I and T, but not, of course, the viscous effects within the superlayer, integration from I to T gives

$$(U_n \Omega_p + \overline{u_n' \omega_p'} - \overline{u_p' \omega_n'})_T = 0.$$

Now,
$$\overline{u_n' \omega_p'} - \overline{u_p' \omega_n'} = \overline{u_n'\left(\frac{\partial u_t'}{\partial x_n} - \frac{\partial u_n'}{\partial x_t}\right)} - \overline{u_p'\left(\frac{\partial u_p'}{\partial x_t} - \frac{\partial u_t'}{\partial x_p}\right)}$$

and, with some manipulation, the right-hand side becomes

$$\frac{\partial}{\partial x_n}\overline{(u_n' u_t')} - \overline{u_t'\left(\frac{\partial u_n'}{\partial x_n} + \frac{\partial u_p'}{\partial x_p} + \frac{\partial u_t'}{\partial x_t}\right)} - \frac{1}{2}\overline{\left(\frac{\partial u_n'^2}{\partial x_t} + \frac{\partial u_p'^2}{\partial x_t} - \frac{\partial u_t'^2}{\partial x_t}\right)} + \frac{\partial \overline{u_p' u_t'}}{\partial x_p}.$$

The second term vanishes by continuity and the last two vanish by the assumed homogeneity in the plane of the superlayer. Hence, the vorticity jump condition is

$$\left[U_n \Omega_p + \frac{\partial}{\partial x_n}\overline{(u_n' u_t')}\right]_T = 0. \qquad (3.1d)$$

Equation (3.1*d*) *requires that there be a discontinuity in mean vorticity across the superlayer proportional to the Reynolds stress gradient at the layer.* This is in accord with the simplified model of Kovasznay (1967) and the experiments of Kovasznay *et al.* (1970).

4. Completion of the problem formulation

To apply the superlayer jump conditions in the problem at hand we assume that $U(y)$ = constant in the irrotational region. The superlayer surface is described by $\tilde{\eta}(x, z, t)$ and we put

$$\tilde{\eta} = \hat{\eta} \exp\left[i(\alpha x + \beta z - \alpha c t)\right]. \qquad (4.1)$$

The solution to (2.8) in the inviscid region must decay as $y \to \infty$, and we have

$$\hat{u}_I = -ib\, e^{-ky}, \quad \hat{v}_I = b\, e^{-ky}, \qquad (4.2)$$

where b is a constant. Equation (2.7) is solved numerically in the turbulent region, and the two regions are coupled by linearized equations derived from (3.1). In deriving these equations care must be taken to transfer properly the conditions from the surface $y = \overline{\eta} + \tilde{\eta}$ to the undisturbed surface $y = \overline{\eta}$ using the usual expansion techniques; the quasi-steady field U_i is replaced by $\overline{u}_i + \tilde{u}_i$ in (3·1) and time averages by phase averages. The following conditions result:

Continuity (3.1*a*):

$$\hat{Q} \equiv [-\hat{v} + i\alpha(U-c)\hat{\eta}]_I = [-\hat{v} + i\alpha(U-c)\hat{\eta}]_T \quad \text{at} \quad y = \overline{\eta}. \qquad (4.3a)$$

Normal momentum (3.1*b*):

$$\hat{p}_I = (\hat{p} - 2D\hat{v}/R_e)_T \quad \text{at} \quad y = \overline{\eta}. \qquad (4.3b)$$

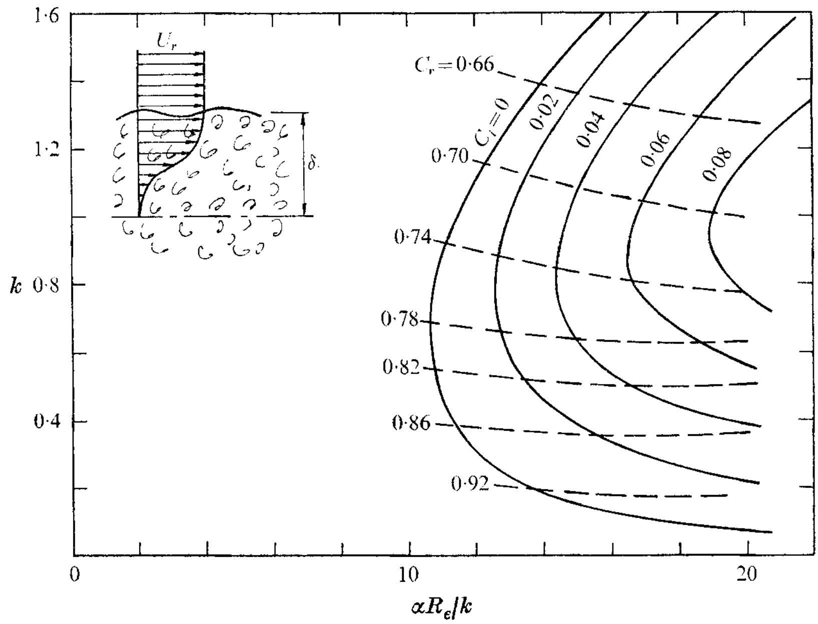

FIGURE 3. Eigenvalues for the cosine wake.

Tangential momentum (3.1*c*):

$$\hat{Q}(U_I - U_T) + \left(\frac{i}{\alpha R_\epsilon}(D^2 + k^2)\hat{v} + \frac{1}{R_\epsilon}(D^2 U)\hat{\eta}\right)_T = 0 \quad \text{at} \quad y = \bar{\eta}. \tag{4.3c}$$

Vorticity (3.1*d*):

$$\left\{(DU)\hat{Q} + \frac{i}{\alpha R_\epsilon}(D^3 + k^2 D)\hat{v} + \frac{1}{R_\epsilon}(D^3 U)\hat{\eta}\right\}_T = 0 \quad \text{at} \quad y = \bar{\eta}. \tag{4.3d}$$

It is of interest to compare these with conditions used previously, and to relate them to experimental data. Townsend's (1966) condition of zero shear results from (4.3*c*) if the entrainment perturbation $\hat{Q}$ is zero and the mean profile is such that $D^2U = 0$ at the undisturbed superlayer. It seems preferable to allow for perturbations in the entrainment rate. Kibens's (1968) data indicate that the velocity discontinuity at the superlayer is negligible and that the velocity gradient in the vicinity of the superlayer is constant, irrespective of the superlayer position. This latter observation is equivalent to the condition

$$(i/\alpha)D^2\hat{v} + (D^2U)\hat{\eta} = 0 \quad \text{at} \quad y = \bar{\eta}. \tag{4.4}$$

Equation (4.4) follows from (4.3*c*) for small longitudinal wavenumbers if $U_T - U_I = 0$ at the superlayer. Kibens's observations is therefore a natural consequence of the tangential momentum condition.

Before Kibens's data were available, we studied a number of wake profiles possessing a velocity discontinuity at the superlayer using the conditions (4.3). These calculations showed instabilities at typical eddy Reynolds numbers, from an analysis along the lines of that of Townsend (1966). These results will not be reported here, as it now seems clear that there is no substantial velocity discontinuity across the superlayer.

In order to study superlayer waves in flow without a discontinuity in velocity at the superlayer, calculations were carried out using a cosine profile:

$$U = \begin{cases} \tfrac{1}{2}[1-\cos(\pi y)] & -1 \leqslant y \leqslant 1, \\ 1 & y < -1, y > 1. \end{cases} \tag{4.5}$$

Here the maximum velocity defect is used as the characteristic velocity and the half-thickness of the turbulent region is the characteristic length (figure 3). $U(y)$ and c are measured relative to the velocity at the centre of the wake. In this normalization the free-stream speed is unity and (4.2) describes the eigenfunctions in the free stream. The profile (4.5) represents what would be observed in a mean wake field which is not diffusing into the free stream. Hence any unstable motions will lead to interaction and to the entrainment of non-turbulent fluid and such instabilities should be predicted by the theory. The instability is a consequence of the vorticity maximum *within* the turbulent region, at the points $y = \pm\frac{1}{2}$ for the profile (4.5). It is a Kelvin–Helmholtz type instability, but not an instability of the superlayer as was suggested by Townsend (1966).

The eigenvalues for eigenfunctions symmetric about the centre-line ($y = 0$) were calculated from numerical solutions of (2.7) satisfying the superlayer matching conditions (4.3). Figure 3 shows the eigenvalues $c(\alpha, \beta, R_\epsilon)$. Note that the critical R_ϵ is about 10 and that a band of unstable oblique waves can exist at $R_\epsilon = 20$. The wave speeds are very close to the maximum speed (unity in this normalization), and the growth rates are substantial. Experiments (Schlichting 1968) indicate that $R_\epsilon \approx 20$ in wake flows, and hence we expect real flows to possess large-scale three-dimensional structures over a band of wavenumbers.

5. Conclusions

Using a Newtonian eddy viscosity to represent the effect of small-scale turbulent motions on large-scale organized waves, we have shown that a turbulent shear flow confined within a smooth uncontorted superlayer would be unstable to a band of large-scale two- and three-dimensional wave disturbances. The instability manifests itself in contortions of the superlayer interface and in wave disturbances that travel downstream at speeds slightly below the free-stream velocity. The amplitude and structure of the resulting finite disturbances could be estimated by nonlinear analysis (Liu 1971), but the extension of the closure model to the finite amplitude problem would be highly questionable. The present model provides the explanation for the large eddy (wave) structure observed in free shear flows and for the existence of waves on the superlayer interface.

The basic flow considered here, i.e. a parallel flow bounded by smooth uncontorted superlayers, is not observed, because of the instability described above. The entrainment resulting from the instability will lead to thickening of the turbulent flow and as the wave disturbance amplitudes grow, their Reynolds stresses will add to those of the background turbulence, causing additional distortions of the flow. The present analysis suggests that (normalized) wavenumbers of the order of $k = 1$ will dominate the large-scale structure. This corresponds to scales of the order of three times the thickness of the 'undisturbed' turbulent core, which might correspond to scales of the same order of magnitude as the thickness of the fully developed turbulent wake. This is in general agreement with experiments. The analysis also suggests that the convection velocity of large-scale disturbances will be about 20 % of the velocity defect below the free-stream speed, and this is also comparable with observations in real flows.

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